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1990 J. Phys. A: Math. Gen. 23 L747S

(http://iopscience.iop.org/0305-4470/23/14/010)

J. Phys. A: Math. Gen. 23 (1990) L747S–L749S. Printed in the UK

LETTER TO THE EDITOR

A new singular behaviour in current distribution of random resistor networks

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Received 8 May 1990

Abstract. We study the singular behaviour occurring in $\partial\alpha(q)/\partial q$ of the multifractal spectrum for the current distribution in random resistor networks. The singular behaviour may be understood as the divergent behaviour of specific heat in thermodynamic phase transition. $\partial\alpha(q)/\partial q$ shows a peak with the width $q_0(L) < q < q_c(L)$, and the height of which diverges as the system size L increases. However the nature of the singular behaviour in $\partial\alpha(q)/\partial q$ is very unusual compared with that of the ordinary phase transition in the following ways. First, the width of the peak becomes broader, rather than narrower, with the increasing system size L, because $q_0(L) \to -\infty$ and $q_c(L) \to 0$ as $L \to \infty$. Second, the singularities at $q = q_0$ and $q = q_c$ are of different types. The singularity at $q = q_0$ is of power-law type such as $\sim L^{d(2-m)/m} \log L$, and the singularity at $q = q_c$ is of logarithmic type such as $\sim (\log L)^{2/m}$, where d is spatial dimension and m is measured to be $\sim 1.633 \pm 0.006$.

Recently, breakdown of multifractal behaviour has drawn much attention [1–4]. For example, moments of the current distribution in random resistor networks (RRN) [5] and moments of the probability distribution in diffusion-limited aggregation (DLA) [6] exhibit multifractal behaviour for positive moments. But the multifractal behaviours in RRN and DLA break down in a certain range of negative moments. The breakdown phenomenon is largely attractive, analogous to a thermodynamic phase transition [2, 3, 7], which will be discussed below.

The multifractal formulation [8] for moments of the current distribution in RRN is defined as

$$M_q \equiv \langle i^q \rangle = \sum_{\log i} n(i) i^q \sim A(q) L^{-\tau(q)} \tag{1}$$

where L is the system size, $n(i)$ is the current distribution, and q represents moment. The multifractal spectrum is analysed by using the conventional formalism such as

$$\alpha = \frac{\mathrm{d}\tau(q)}{\mathrm{d}q} \qquad f = q\alpha - \tau. \tag{2}$$

The mathematical form of (1) looks similar to a thermodynamic partition function, which leads to the analogy that M_q corresponds to a partition function; q corresponds to the inverse temperature: $\tau(q)$ and $\alpha(q)$ represent a free energy and an internal energy respectively; specific heat is associated with $\partial\alpha(q)/\partial q$.

0305-4470/90/14747S+03$03.50

The multifractality means that $\tau(q)$ and $\alpha(q)$ are independent of L, and which occurs in positive moments. However, that $\tau(q)$ and $\alpha(q)$ are independent of L does not appear in a whole range of moments. But they depend on L in a certain range of negative moments, which implies the occurrence of breakdown of the multifractal behaviour. In a previous work [9], by using the extreme statistics idea [10], it was found that there exist distinct critical values q_c, q_0 ($q_c > q_0$) such that only for $q > q_c$, the size-independent behaviour appears. Physically q_0 is originated from the ultraviolet cut-off, which is due to the smallest current over all configurations. The size-dependent behaviour of $q_c(L)$ and $q_0(L)$ was derived explicitly, in which $q_0(L) \to -\infty$ and $q_c(L) \to 0$ as $L \to \infty$. Accordingly, the phase transition occurring at $q = q_0(L)$ was regarded as a trivial one in the thermodynamic limit, $L \to \infty$. However the phase transition occurring at $q = q_c(L)$ was not examined in the previous work. This transition is worth noting by showing that the specific heat also diverges at finite value of moments $q = q_c$ even in the thermodynamic limit. Furthermore the origin of the singularity at $q = q_c$ is quite different from the one at $q = q_0$. While the latter is due to finite size scaling, the former is due to the intrinsic property of the distribution function. Thus the main purpose of this letter is to examine a new singular behaviour at $q = q_c$ as a complement of the previous work [9].

We begin by recalling the explicit form of the size-dependent behaviour for $q_0(L)$ and $q_c(L)$ derived in [9],

$$q_0(L) \sim -L^{d(m-1)/m}/\log L \tag{3}$$

and

$$q_c(L) \sim -(\log L)^{-1/m} \tag{4}$$

where m is measured to be $m = 1.633 \pm 0.006$ and d is spatial dimension. The explicit form of $\tau(q)$, $\alpha(q)$, and $f(\alpha)$ is also derived as follows. For $q < q_0(L)$

$$\begin{aligned} \tau(q) &\sim -L^{d/m}(-q) \\ \alpha(q) &\sim L^{d/m} \\ f(\alpha) &\sim \text{constant} \end{aligned} \tag{5}$$

and for $q_0(L) < q < q_c(L)$

$$\begin{aligned} \tau(q) &\sim -(\log L)^{1/(m-1)}(-q)^{m/(m-1)} \\ \alpha(q) &\sim (\log L)^{1/(m-1)}(-q)^{1/(m-1)} \\ f(\alpha) &\sim -\alpha^m/\log L. \end{aligned} \tag{6}$$

But when $q > q_c(L)$, $\tau(q)$ and $\alpha(q)$ are independent of the system-size L.

For the interval $q_0(L) < q < q_c(L)$, we can easily obtain the size-dependent behaviour of $\partial\alpha(q)/\partial q$, which corresponds to the specific heat in the analogy, as

$$\frac{\partial\alpha(q)}{\partial q} \sim (\log L)^{1/(m-1)}(-q)^{(2-m)/(m-1)}. \tag{7}$$

Therefore the specific heat increases with increasing L when $m > 1$, and eventually it diverges as $L \to \infty$. Since the specific heat equals to zero in the regions, $q > q_c$ and $q < q_0$ and diverges in the interval, two singular behaviours occur around $q = q_c$ and $q = q_0$. By inserting (3) and (4) into (7), we can find the two singularities. It turns out that the nature of the singularities are quite different from each other. One singularity is of power-law type such as $\sim L^{d(2-m)/m} \log L$ at $q \to q_0$, and another is logarithmic such as $\sim(\log L)^{2/m}$ at $q \to q_c$. This asymmetric behaviour is very unusual compared with that of the ordinary thermodynamic phase transition. Furthermore the width of the peak, $q_0 < q < q_c$, becomes broader instead of narrower as the system size L increases. This type of phase transition is firstly observed in this work. So far, the above discussion is based on RRN of which the Weibull constant is $1 < m < 2$. However if $m = 2$, then the power-law singularity would disappear and would change to the logarithmic one. Thus the singularities of each side of the peak reduce to be of the same type. Finally it is worth mentioning that the broadening behaviour of the width appears only when $m > 1$.

In summary, we have considered the unusual phase transition occurring in the multi-fractal formulation for the current distribution in random resistor networks. The specific heat, $\partial\alpha(q)/\partial q$, shows a peak which increases rapidly across $q \to q_c(L)$ and decreases rapidly across $q \to q_0(L)$ as q decreases. The width of the peak becomes broader as the system size increases, which is anomalous to the usual behaviour that the width becomes narrower. Moreover, the nature of singularities on each side of the peak are quite different from each other. One is of power-law type with logarithmic correction and another is of logarithmic type.

One of authors (BK) acknowledges support from the Regents of University of California. Also, JL is grateful for the useful discussions with S Havlin, S Schwarzer and H E Stanley. The Center for Polymer Studies is supported in part by grants from the NSR and ONR. This financial support is gratefully acknowledged.

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